SUPERFRACTALTHING MATHS

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The standard Mandelbrot set equation can be written as:
$X_{n+1} = X_n^2 + X_0$

Where the complex number X_0 is in the Mandelbrot set if X_n stays finite as n tends to infinity. Traditionally, when creating mandelbrot set images, points in the Mandelbrot set are coloured black, and points outside have a colour generated from the minimum value of n required to give the complex number X_n a magnitude of more than 2.

Consider another point given by Y_0
$Y_{n+1} = Y_n^2 + Y_0$

The difference between these two points at a given iteration is given by Δ_n such that

$\Delta_n = Y_n - X_n$

Then

$\Delta_{n+1} = Y_{n+1} - X_{n+1}$
$\Delta_{n+1} = (Y_n^2 + Y_0) - (X_n^2 + X_0)$
$\Delta_{n+1} = ((X_n + \Delta_n)^2 + X_0 + \Delta_0) - (X_n^2 + X_0)$
$\Delta_{n+1} = (X_n^2 + 2X_n\Delta_n + \Delta_n^2 + X_0 + \Delta_0) - (X_n^2 + X_0)$

$$\Delta_{n+1} = 2X_n\Delta_n + \Delta_n^2 + \Delta_0 \qquad \textbf{(1)}$$

Equation (1) is important, as all the numbers are 'small', allowing it to be calculated with hardware floating point numbers. So if we know all the values X_n, we can use this equation to calculate Y_n without having to use arbitrary precision calculations.

Let $\delta = \Delta_0$

$\Delta_1 = 2X_0\delta + \delta^2 + \delta = (2X_0 + 1)\delta + \delta^2$
$\Delta_2 = (4X_1X_0 - 2X_1 - 1)\delta + ((x_0 - 1)^2 + 2X_1)\delta^2 + (4X_0 - 2)\delta^3 + o(\delta^4)$

Let $\Delta_n = A_n\delta + B_n\delta^2 + C_n\delta^3 + o(\delta^4) \qquad \textbf{(2)}$

Then

$A_{n+1} = 2X_nA_n + 1$ **(3)**
$B_{n+1} = 2X_nB_n + A_n^2$ **(4)**
$C_{n+1} = 2X_nC_n + 2A_nB_n$ **(5)**

Now we can apply equations (3), (4) and (5) iteratively to calculate the coefficents for equation (2). Equation (2) can then be used to calculate the value for the n^{th} iteration for all the points around X_0. The approximation should be good as long as the δ^3 term has a magnitude significantly smaller then the δ^2 term.

Using equations (1) and (2) means that the time taken rendering Mandelbrot images is largely independent of depth and iteration count, and mainly depends on the complexity of the image being created.